George M. Bergman
61 Evans Hall

Fall 1997, Math H1A

Final Exam

18 December, 1997
12:30-3:30 PM

1. (30 points, 5 points apiece) Compute each of the following.

(a) $d/dx\,(\ln(\ln x))$.

(b) $\lim_{x\to\pi/4}\,((\tan x)-1)/((\cos^2 x)-½)$.

(c) $\lim_{x\to 0^-}\,(\tan x)/((\cos^2 x)-1)$.

(d) An expression for the "telescoping sum" $\Sigma_{i=5}^{1{,}000} f(i+1)-f(i)$ in which cancelling summands have been dropped.

(e) $\lim_{n\to+\infty}\,(\Sigma_{i=1}^{n}\,10^{i/n})/n$.

(f) $\int x\cos(x^2/3)\,dx$.

2. (18 points) The Riemann integral of a function f was defined by the formula

$$\int_a^b f(x)\,dx \;=\; \lim_{\|P\|\to 0}\,\Sigma_{i=1}^{n}\,f(x_i^*)\,\Delta x_i.$$

(a) (9 points) In the above formula, P denotes a *partition* of $[a,b]$, $\|P\|$ denotes its *norm*, and the Δx_i denote certain related numbers. Define each of these:
A *partition* P of $[a,b]$ means

Given such a partition P, we define $\Delta x_i =$

For such a partition P, its *norm* $\|P\|$ means

(b) (9 points) Now complete the sentence below to give a precise ("ε-δ") definition of the limit expression in the above definition of the Riemann integral. (You do not, of course, have to explain Σ-notation. On the other hand, your definition should make clear what the symbol x_i^* refers to.)

If f is a function on an interval $[a,b]$, and L is a real number, we write $\lim_{\|P\|\to 0}\,\Sigma_{i=1}^{n}\,f(x_i^*)\,\Delta x_i \;=\; L$ if

3. (15 points) (a) (5 points) Suppose f is a continuous function on $(0, +\infty)$, and a and b are positive real numbers such that $f(ax) = bf(x)$ for all $x > 0$. Show that if F is any antiderivative of f, there will exist a constant k such that $F(ax) = abF(x) + k$ for all $x > 0$.

(b) (5 points) Letting $f(x) = 1/x^2$, and a be an arbitrary positive real number, give a value of b that makes the equation $f(ax) = bf(x)$ hold. Give the general antiderivative F of this function f, and for each such antiderivative find a constant k satisfying the equation given in (a).

(c) (5 points) Do the same for $f(x) = 1/x$.

4. (20 points, 5 points apiece) Give an example of each of the following. You do *not* have to prove that your examples have the properties asked for. When asked to give a function with a given domain $[a, b]$, you may give a function with larger natural domain, understanding it to be restricted to the domain asked for.

(a) A function f on $[0,1]$ having no maximum value.

(b) A continuous function f on $[1, 10]$ which has a local maximum that is not an absolute maximum.

(c) A continuous function f on $[0, 5]$ which is concave upward on $[0, 3]$ and concave downward on $[3, 5]$.

(d) A one-to-one function f with domain $[0,1]$ and range $[3, 5]$, and the function g inverse to f.

5. (7 points) Given that $d/dx \tan x = \sec^2 x$, get a formula for $\int_a^b \sec^2 x \, dx$, and state conditions on a and b under which your formula is valid. (For full credit, you should give the most general such conditions.)

6. (10 points) In the last reading, we defined the exponential function exp as the inverse of the natural logarithm function ln. Using this definition, prove the identity $\exp(x+y) = \exp(x)\exp(y)$. The proof uses one or more properties of the natural logarithm function; state explicitly the properties you use. (You are not asked to prove those properties of the logarithm, or to say anything about how the logarithm function is defined.)